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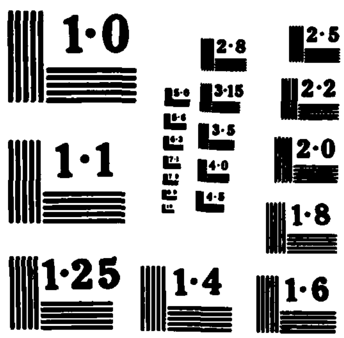
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INFERENCE ON THE RANKS OF THE CANONICAL CORRELATION MATRICES FOR ELLIPTICALLY SYMMETRIC POPULATIONS

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SECURITY CLASSIFICATION OF THIS PAGE

ADA 158 268

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: Unclassified	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY: ---	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE: N/A	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 85-0550
6a. NAME OF PERFORMING ORGANIZATION: University of Pittsburgh; 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION: AFOSR
6c. ADDRESS (City, State and ZIP Code): 515 Thackery Hall, Pittsburgh, PA 15260	7b. ADDRESS (City, State and ZIP Code): Bldg. 410, Bolling AFB, D.C. 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR; 8b. OFFICE SYMBOL (If applicable): NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F49620-85-C-0008
8c. ADDRESS (City, State and ZIP Code): Bldg. 410, Bolling AFB, D.C. 20332-6448	10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2304	A5	

11. TITLE (Include Security Classification): Inference on the Ranks of the Canonical Correlation Matrices for Elliptically SYMMETRIC Populations

12. PERSONAL AUTHOR(S): P. R. Krishnaiah, J. Lin and L. Wang

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Technical	FROM ____ TO ____	May 1985	27

16. SUPPLEMENTARY NOTATION

17. COSATI CODES		
FIELD	GROUP	SUB. GR.
XXXXX	XXXXXXXXX	XXX

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number): Asymptotic distributions, complex distributions, canonical correlations, elliptical distribution, time series

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☒ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION: Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL: Brian W. Woodruff, Maj, USAF	22b. TELEPHONE NUMBER (Include Area Code): (202)767-5027; 22c. OFFICE SYMBOL: NM

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE.

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ABSTRACT

In this paper, the authors considered the likelihood ratio tests and some other tests for the ranks of the canonical correlation matrices when the underlying distributions are real and complex elliptically symmetric distributions. Also, asymptotic joint distributions of the eigenvalues of the sample canonical correlation matrices are derived under the assumptions mentioned above regarding the underlying distributions. Finally, applications of tests for the rank of the complex canonical correlation matrix in the area of time series in the frequency domain are discussed.

Key Words: asymptotic distributions, complex distributions, canonical correlations, elliptical distribution, and time series.

INFERENCE ON THE RANKS OF THE CANONICAL CORRELATION MATRICES FOR ELLIPTICALLY SYMMETRIC POPULATIONS

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May 1985

Technical Report No. 85-14

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[1]The work of this author was supported by the Air Force Office of Scientific Research under Contract F49620-85-C-0008. The United States Government is authorized to reproduce and distribute reprints for governmental purposes not withstanding any copyright notation herein.

1. INTRODUCTION

In studying the relationships between two sets of variables, it is of interest to test for the rank of the canonical correlation matrix. The purpose of such analysis is to find a small number of pairs of linear combinations of original variables which would best explain the relationship between the two sets of original variables. Bartlett (1947) proposed a procedure to test for the rank of the canonical correlation matrix and derived asymptotic distribution of the associated test statistic. The above procedure is the likelihood ratio test (LRT) procedure (see Fujikoshi (1974)). Fujikoshi (1978) derived asymptotic nonnull distribution of a function of the sample canonical correlations whereas Krishnaiah and Lee (1979) derived asymptotic joint distribution of certain functions of the canonical correlations when the population canonical correlations have multiplicity and none of the canonical correlations are zero. The work reported above was done under the assumption that the underlying distribution is multivariate normal. But, situations arise often when it is unrealistic to assume that the underlying distribution is multivariate normal. In these situations, it is of interest to study the robustness of the standard test procedures for the rank and derive the LRT procedure under non-normal assumptions. It is also of interest to study the distributions of various test statistics for the rank of the canonical correlation matrix using a robust estimate of the covariance matrix. For some discussions on the robust estimates of the covariance matrix, the reader is referred to Devlin, Gnanadesikan and Kettenring (1975) and Maronna (1976). Murihead and Waternaux (1980) derived the asymptotic joint distribution of the sample canonical correlations when the underlying distribution is elliptically symmetric and the population canonical correlations are distinct and nonzero. Fang and Krishnaiah (1982) derived joint distributions of certain functions of the sample canonical correlations for large samples when the underlying distribution is non-normal and the population canonical correlations are nonzero and have multiplicity. The results obtained by Fang and Krishnaiah include the term of order $n^{-\frac{1}{2}}$ also where (n+1) is the sample size. The object of this paper is to derive the LRT procedures and derive the asymptotic distributions of certain statistics for the ranks of the canonical correlation matrices when the underlying distributions are real and complex elliptically symmetric.

In Section 2, we derive the LRT procedures for the rank of the canonical correlation matrices when the joint distributions of all the observations is real and complex elliptically symmetric. The test statistics and their distributions in the above cases are the same as when the underlying distributions are real and complex multivariate normal. The above situations are different from those when the underlying distributions are real and complex elliptically symmetric. In Section 3, we derive asymptotic joint distributions of the canonical correlations in the latter situations. A discussion of various test procedures for the rank of the canonical correlation matrix is given in Section 4. Applications of these procedures in the area of time series in the frequency domain are also discussed in this section.

2. LRT PROCEDURES FOR TESTING THE RANK OF THE CANONICAL CORRELATION MATRIX IN THE REAL AND COMPLEX CASES

Let X: $n\times p$ be a random matrix whose elements are distributed as elliptically symmetric with density function (see Kelker (1970)) given by

$$f(X) = |\Sigma|^{-n/2} h(\mathrm{tr}\ \Sigma^{-1}(X'X)) \tag{2.1}$$

where h is a strictly decreasing and differentiable function ,

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \quad X = (X_1\ X_2)$$

Σ_{ij} is of order $p_i\times p_j$, X_1 is of order $n\times p_1$, X_2 is of order $n\times p_2$ and $p = p_1+p_2$. We need the following lemma in the sequel to derive the LRT procedure for the rank of Σ_{12}.

Lemma 2.1. Let $\hat{S} = Q^{-1/2}P'PQ^{-1/2}$, P is a $n\times p$ matrix, Q is a positive definite matrix such that the rank of $\hat{S}$ is at least r. Also, let $\pi_r(\underset{\sim}{0})$ denote the set of matrices M such that $M = GF$ where G: $n\times r$ is such that $|G'G| \neq 0$ and $FF' = I_r$. Then, the eigenvalues of $S(M) = Q^{-1/2}(P-M)'(P-M)Q^{-1/2}$ are minimized simultaneously with respect to $M \in \pi_r(\underset{\sim}{0})$ if and only if

$$M = P Q^{-1/2} V_r' V_r Q^{1/2} \tag{2.2}$$

where the rows of V_r consist of normalized eigenvectors corresponding to the first r largest eigenvalues of $\hat{S}$. The minimum values of eigenvalues of S(M) are given by ϕ_{r+i} where $\phi_1 \geq \ldots \geq \phi_p$ are the eigenvalues of $\hat{S}$ and $\phi_j = 0$ for $j > p$.

For a proof of the above lemma, the reader is referred to Fujikoshi (1974).

We now consider the problem of testing the hypothesis H_s against H_t where

$$H_s:\ \mathrm{rank}(\Sigma_{12}) = s.$$

The likelihood function is given by

$$L = \{|\Sigma_{22}|\ |\Sigma_{11.2}|\}^{-n/2}\ h(\mathrm{tr}\ \Sigma_{11.2}^{-1}(X_1 - X_2B)'(X_1 - X_2B)$$
$$+ \mathrm{tr}\ \Sigma_{22}^{-1}X_2'X_2) \tag{2.3}$$

where $B = \Sigma_{22}^{-1}\Sigma_{21}$. It is seen (see Anderson and Fang (1982)) that

$$\max_{\Sigma_{11.2},\Sigma_{22}} L = \{\lambda_{max}(h)\}^{-np/2}\ h(p/\lambda_{max}(h))$$
$$(|(X_1 - X_2B)'(X_1 - X_2B)|\ |(X_2'X_2)|)^{-n/2} \tag{2.4}$$

where $\lambda_{max}(h)$ is a constant depending upon the form of $h(\cdot)$. Also

$$|(X_1 - X_2B)'(X_1 - X_2B)|$$
$$= |S_{11.2}||I + S_{11.2}^{-1/2}(S_{22}^{-1/2}S_{21} - S_{22}^{1/2}B)'(S_{22}^{-1/2}S_{21} - S_{22}^{1/2}B)S_{11.2}^{-1/2}| \tag{2.5}$$

and $\mathrm{rank}(\Sigma_{12}) = \mathrm{rank}(S_{22}^{1/2}B)$. Here $S_{11} = X_1'X_1$, $S_{22} = X_2'X_2$ and $S_{12} = X_1'X_2$. Applying Lemma 2.1, we obtain the following expression for the maximum likelihood estimate of B when the rank of B is s:

$$\hat{B} = S_{22}^{-1}S_{21}S_{11.2}^{1/2}V_s'V_sS_{11.2}^{-1/2} \tag{2.6}$$

where V_s consists of the normalized eigenvectors corresponding to the first s largest eigenvalues of $S_{11.2}^{1/2}S_{12}S_{22}^{-1}S_{21}^{-1}$. We know that $V_{p_1}'V_{p_1} = I_{p_1}$ and so

$$\hat{B} = S_{22}^{-1}S_{21} \tag{2.7}$$

when $s = p_1$.

By Lemma 2.1, we obtain

$$\max_{\Sigma_{12}\in\pi_s(\underset{\sim}{0})}\ \max_{\Sigma_{11.2},\Sigma_{22}} L = \{\lambda_{max}(h)\}^{-np/2} h(p/\lambda_{max}(h))$$

$$\times\ |S_{11.2}|\{(1+\ell_{s+1})\ldots(1+\ell_{p_1})\}^{-n/2} \tag{2.8}$$

for $s < p_1$ where $\ell_1 \geq \ldots \geq \ell_{p_1}$ are the eigenvalues of $S_{11.2}^{-1}S_{12}S_{22}^{-1}S_{21}$. Also,

$$\max_{\Sigma_{12}\in\pi_{p_1}(\underset{\sim}{0})}\ \max_{\Sigma_{11.2},\Sigma_{22}} L = \{\lambda_{max}(h)\}^{-np/2} h(p/\lambda_{max}(h))\ |S_{11.2}|\,. \tag{2.9}$$

The LRT statistic for testing H_s against H_t $(t > s)$ is given by

$$\tau_{st} = \prod_{j=s+1}^{t} (1-r_j^2)^{n/2} \tag{2.10}$$

where $r_1^2 \geq \ldots \geq r_{p_1}^2$ are the eigenvalues of $S_{11}^{-1}S_{12}S_{22}^{-1}S_{21}$. When the underlying distribution is multivariate normal, the above derivation was given in Fujikoshi (1974).

We will now consider the LRT procedure for the rank of the canonical correlation matrix when the joint distribution of the observations is complex elliptically symmetric. In this case, the joint distribution of the observations is given by

$$f(Z) = \frac{2^{np}}{|\Sigma|^n} h(2\,\mathrm{tr}\,\Sigma^{-1}Z'\bar{Z}) \tag{2.11}$$

where Z is Hermitian positive definite matrix and $\bar{Z}$ denotes the complex conjugate of Z. Complex elliptically symmetric distribution was introduced by Krishnaiah and Lin (1984). When $h(y) = \exp(-y/2)$, the rows of Z are distributed independently as complex multivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix Σ. We need the following lemma which can be proved along the same lines as Lemma 2.1:

<u>Lemma 2.2</u>. Let $\hat{S} = Q^{-1/2}P'PQ^{-1/2}$ where P: $u\times p$ is a complex matrix, Q is Hermitian positive definite matrix. Also, let $M = GF$ where G: $u\times r$ is a complex matrix with $|\bar{G}'G| \neq 0$ and $F\bar{F}' = I_r$. Then the eigenvalues of $S(M) = Q^{-1/2}(P-M)'(\overline{P-M})Q^{-1/2}$ are

minimized simultaneously with respect to $M \in \pi_r(\underset{\sim}{0})$ if and only if

$$M = P\bar{Q}^{-1/2}\,\bar{V}_r' V_r \bar{Q}^{1/2} \tag{2.12}$$

where the rows of V_r consist of normalized eigenvectors corresponding to the first r largest eigenvalues of $\hat{S}$. The minimum values of the eigenvalues of S(M) are given by ϕ_{r+i} where $\phi_1 \geq \ldots \geq \phi_p$ are the eigenvalues of $\hat{S}$ and $\phi_j = 0$ for $j \geq p$.

Consider the likelihood function

$$L = \frac{2^{np}}{[|\Sigma_{22}||\Sigma_{11.2}|]^n}\, h(2\,\mathrm{tr}\,\Sigma_{11.2}^{-1}(Z_1 - Z_2B)'(\overline{Z_1 - Z_2B}) + 2\,\mathrm{tr}\,\Sigma_{22}^{-1}Z_2'\bar{Z}_2) \tag{2.13}$$

where $B = \bar{\Sigma}_{22}^{-1}\bar{\Sigma}_{21}$. As in the real case, we can show that

$$\max_{\Sigma_{11.2},\Sigma_{22}} L = \frac{2^{np}}{\left\{\lambda^*_{max}(h)\right\}^{np}}\, h(2p/\lambda^*_{max}(h))$$

$$\times\ (|(Z_1 - Z_2B)'(\overline{Z_1 - Z_2B})||Z_2'\,\bar{Z}_2|)^{-n} \tag{2.14}$$

where $\lambda^*_{max}(h)$ is a constant. But

$$|(Z_1 - Z_2B)'(\overline{Z_1 - Z_2B})|$$

$$= |W_{11.2}||I + W_{11.2}^{-1/2}(W_{22}^{-1/2}W_{21} - W_{22}^{1/2}B)'\overline{(W_{22}^{-1/2}W_{21} - W_{22}^{1/2}B)W_{11.2}^{-1/2}}| \tag{2.15}$$

and $\mathrm{rank}(\Sigma_{12}) = \mathrm{rank}(W_{22}^{1/2}\bar{\Sigma}_{22}^{-1}\bar{\Sigma}_{21})$. Here $W_{11} = Z_1'\bar{Z}_1$, $W_{22} = Z_2'\bar{Z}_2$ and $W_{12} = Z_1'\bar{Z}_2$. So, by Lemma 2.2, we obtain

$$\max_{\Sigma_{12}\in\pi_s(\underset{\sim}{0})}\ \max_{\Sigma_{11.2},\Sigma_{22}} L = \frac{2^{np}}{\left\{\lambda^*_{max}(h)\right\}^{np}}\, h(2p/\lambda^*_{max}(h))$$

$$\times\ |W_{11.2}| \prod_{j=s+1}^{p_1} (1-\theta_j)^n \tag{2.16}$$

where $\theta_1 \geq \ldots \geq \theta_{p_1}$ are the eigenvalues of the matrix $W_{11}^{-1}W_{12}W_{22}^{-1}W_{21}$.

Also,

$$\hat{B} = W_{22}^{-1}W_{21}W_{11.2}^{1/2}\bar{V}_s^{*\prime}V_s^{*}W_{11.2}^{-1/2} \tag{2.17}$$

where V_s^* consists of the normalized eigenvectors corresponding to the first s largest eigenvalues of $W_{11.2}^{1/2}W_{12}W_{22}^{-1}W_{21}^{-1}$. So, the LRT statistic for testing H_s against H_t is

$$\tau_{st}^* = \prod_{j=s+1}^{t} (1-\theta_j)^n. \tag{2.18}$$

So, the LRT statistic for testing H_s against the alternative that Σ_{12} is of full rank is

$$\tau_{sp_1}^* = \prod_{j=s+1}^{p_1} (1-\theta_j)^n. \tag{2.19}$$

Similarly, the LRT procedure for $\Sigma_{12} = 0$ against the hypothesis that the rank is t is given by

$$\tau_{0t}^* = \prod_{j=1}^{t} (1-\theta_j)^n. \tag{2.20}$$

Here we note that Anderson and Fang (1982) derived the LRT procedure for $\Sigma_{12} = 0$ when the underlying distribution is real elliptically symmetric.

3. ASYMPTOTIC JOINT DISTRIBUTION OF THE CANONICAL CORRELATIONS

We will now derive the joint asymptotic distributions of the canonical correlations under two situations. In the first situation, we assume that the density of $Z' = [\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n]$: $p\times n$ is of the form

$$f(X) = |\Sigma|^{-n/2} h(\mathrm{tr}\ \Sigma^{-1} XX'). \tag{3.1}$$

In the second situation, we assume that $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n$ are distributed independently and identically with a common density

$$f(\underset{\sim}{x}) = |\Sigma|^{-1/2} h(\mathrm{tr}\ \Sigma^{-1} \underset{\sim}{x}\underset{\sim}{x}'). \tag{3.2}$$

Without loss of generality, we assume that

$$\Sigma = \begin{pmatrix} I_{p_1} & \tilde{P} \\ \tilde{P}' & I_{p_2} \end{pmatrix} \tag{3.3}$$

where $p_1 < p_2$,

$$\tilde{P} = \begin{bmatrix} \rho_1' & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \rho_2' & \ldots & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot\cdot\cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot \\ 0 & 0 & \ldots & \rho_{p_1}' & 0 & \ldots & 0 \end{bmatrix}. \tag{3.4}$$

In addition, we assume that

$$\begin{aligned} \rho_1' = \ldots &= \rho_{\mu_1}' = \rho_1 \\ \rho_{\mu_1+1}' = \ldots &= \rho_{\mu_1+\mu_2}' = \rho_2 \\ &\vdots \\ \rho_{\mu_1+\ldots+\mu_{s-1}+1}' = \ldots &= \rho_{\mu_1+\ldots+\mu_s}' = \rho_s \\ \rho_{s+1}' = \ldots &= \rho_{p_1}' = 0 \end{aligned} \tag{3.5}$$

where $\rho_1 > \ldots > \rho_s > 0$. When the underlying distribution has density (3.1), it is known (e.g., see Cambanis, Hwang and Simons (1981) Anderson and Fang (1982)) that

$$X = RUA \tag{3.6}$$

where $R^2 = \mathrm{tr}\, X'X$, and vecU has uniform distribution on the unit sphere independent of R^2, $\Sigma = A'A$, $A = (A_1\ A_2)$ and A_1 is of order $p \times p_1$. Here vecU denotes the column vector formed by putting each column of U one below the other. Then, $X_1 = RUA_1$ and $X_2 = RUA_2$. So, $S_{11}^{-1}S_{12}S_{22}^{-1}S_{21}$ is independent of R. Also, U can be expressed as $U = V/\mathrm{tr}V'V$ where the rows of V are distributed independently as multiviarate normal with mean vector $\underset{\sim}{0}$ and covariance matrix Σ. Hence, the distribution of $S_{11}^{-1}S_{12}S_{22}^{-1}S_{21}$ is independent of the particular form of the underlying distribution as long as the underlying distribution belongs to the family of elliptically symmetric distributions. We will now derive the joint asymptotic distribution of the eigenvalues of the sample canonical matrix when the observations $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n$ have a common density (3.2).

We need the following lemmas in the sequel.

<u>Lemma 3.1</u>. If $\underset{\sim}{x}' = (x_1, \ldots, x_p)$ is real elliptically symmetric distribution with finite fourth moments, and density given by $|\Sigma|^{-1/2}h(\mathrm{tr}\Sigma^{-1/2}\underset{\sim}{x}\underset{\sim}{x}')$, then

$$E(x_i x_j x_k x_\ell) = 4\phi''(0)(\sigma_{ij}\sigma_{k\ell} + \sigma_{ik}\sigma_{j\ell} + \sigma_{i\ell}\sigma_{kj}) \tag{3.7}$$

where $\phi''(\cdot)$ denotes the second derivative of $\phi(\cdot)$ and $\Sigma = (\sigma_{ij})$.

Now, let $S_{11} = (a_{ij})$, $S_{12} = (b_{ig})$ and $S_{22} = (c_{gh})$. Also, let

$$
\begin{aligned}
a_{ii} &= -2n\phi'(0) - 2\sqrt{n}\,\phi'(0)u_{ii} \\
a_{ij} &= -2\sqrt{n}\,\phi'(0)u_{ij} \qquad i \neq j \\
b_{ii} &= -2n\phi'(0)\rho_i - 2n\phi'(0)v_{ii} \\
b_{ij} &= -2\phi'(0)\sqrt{n}\,v_{ig} \qquad (i \neq g) \\
c_{gg} &= -2n\phi'(0) - 2\sqrt{n}\,w_{gg}\phi'(0) \\
c_{gh} &= -2\phi'(0)\sqrt{n}\,w_{gh} \quad (g \neq h)
\end{aligned}
\tag{3.8}
$$

for $i,j = 1,2,\ldots,p_1$ and $g,h = 1,2,\ldots,p_2$.

<u>Lemma 3.2</u>. The asymptotic joint distribution of u_{ij}'s, v_{tu}'s and w_{gh}'s is $p(p+1)/2$ variate normal with mean vector $\underset{\sim}{0}$ and second moments given in the following statements:

(i) any v_{ij} and w_{th} which has at least one suffix number $> p_1$ is uncorrelated with all the others;

(ii) any member of one of the sets (u_{ii}, v_{ii}, w_{ii}) $(i = 1,\ldots,p)$, (u_{ij}, v_{ij}, w_{ij}), $(i,j = 1,\ldots,p_1;\ i \neq j)$ is uncorrelated with all the members of all the other sets;

(iii) for $i,j = 1,\ldots,p$, we have

$$
\begin{aligned}
&E\,u_{ii}^2 = E\,w_{ii}^2 = 3(\kappa+1) - 1 \\
&E\,v_{ii}^2 = (\kappa+1) + \rho_i^2[2(\kappa+1)-1] \\
&E(u_{ii}w_{ii}) = \kappa + 2(\kappa+1)\rho_i^2 \\
&E(u_{ii}v_{ii}) = E(w_{ii}v_{ii}) = [3(\kappa+1)-1]\rho_i \\
&E(u_{ij}^2) = E(v_{ij}^2) = E(w_{ij}^2) = \kappa+1 \quad (i \neq j) \\
&E(u_{ij}w_{ij}) = E(v_{ij}v_{ji}) = (\kappa+1)\rho_i\rho_j \quad (i \neq j) \\
&E(u_{ij}v_{ij}) = E(w_{ij}v_{ij}) = (\kappa+1)\rho_j \quad (i \neq j)
\end{aligned}
\tag{3.9}
$$

where

$$\kappa = \frac{\phi''(0) - [\phi'(0)]^2}{[\phi'(0)]^2} .$$

Proof: The asymptotic normality follows from well known central limit theorem. We prove only (iii). From (3.8) we have

$$E\, u_{ii}^2 = \frac{1}{4n[\phi'(0)]^2} \mathrm{Var}(a_{ii}).$$

Also,

$$\mathrm{Var}\, a_{ii} = \sum_{\ell=1}^{n} \mathrm{Var}\, x_{1\ell i}^2 \quad .$$

$$= 4[\phi'(0)]^2 n\, [3(\kappa+1)-1]$$

where $X_t = (x_{t\ell i})$ for $t = 1,2$. So,

$$E\, u_{ii}^2 = 3(\kappa+1) - 1.$$

Similarly,

$$E\, u_{ii} w_{ii} = \frac{1}{4[\phi'(0)^2 n} [E\, a_{ii} c_{ii} - 4[\phi'(0)]^2 n^2]$$

and

$$E\, a_{ii} c_{ii} = \sum_{k,\ell=1} E\, x_{1ki}^2 x_{2\ell i}^2$$

$$= 4[\phi'(0)]^2 n[\kappa + 2\rho_i^2(\kappa+1)] + 4[\phi'(0)]^2 n^2 .$$

So

$$E\, u_{ii} w_{ii} = \kappa + 2\rho_i^2(\kappa+1).$$

In the same manner we can prove other relations.

Lemma 3.3. Let $g_n(x)$ be a sequence of K-degree polynomials with roots $X_1^{(n)}, \ldots, X_K^{(n)}$ for each n, and let g(x) be a k polynomial wtih roots $x_1, \ldots, x_k$, $k \leq K$. If

$g_n(x) \to g(x)$ as $n \to \infty$, then after suitable rearrangement of $x_1^{(n)},\ldots,x_K^{(n)}$ we have $x_j^{(n)} \to x_j$, $j = 1,2,\ldots,k$ and $|x_j^{(n)}| \to \infty$, $j = k+1,\ldots,K$.

For a proof of the above lemma, the reader is referred to Bai (1984). We have the following theorem.

<u>Theorem 3.1</u>. Let the population canonical correlations be $\rho_1',\ldots,\rho_{p_1}'$ which satisfy (3.5). Also, let the sample canonical correlations be $r_1,\ldots,r_{p_1}$ where $r_1 \geq \ldots \geq r_{p_1} > 0$. In addition, let

$$\eta_i = n^{1/2}(1 - \rho_i'^2)^{-1}\,(r_i - \rho_i') \tag{3.9a}$$

for $i = 1,2,\ldots,p_1$. Then the limiting distribution of $\eta_1,\ldots,\eta_{p_1}$ is represented by the density function

$$f(\eta_1,\ldots,\eta_{\mu_1})f(\eta_{\mu_1+1},\ldots,\eta_{\mu_1+\mu_2})\ldots f(\eta_{\mu_1+\ldots+\mu_{s-1}},\ldots,\eta_s)\; f_1(\eta_{s+1},\ldots,\eta_{p_1}) \tag{3.10}$$

where

$$\begin{aligned} f(x_1,\ldots,x_m) &= 2^{-m(m+1)/2}(\kappa+1)^{-m/2}\Big(\prod_{i=1}^{m}\Gamma\big(\tfrac{1}{2}i\big)\Big)^{-1}\Big\{\prod_{i=1}^{m}\prod_{j=i+1}^{m}\Big(\frac{x_i}{\sqrt{\kappa+1}} - \frac{x_j}{\sqrt{\kappa+1}}\Big)\Big\} \\ &\quad \times \exp\Big\{-\frac{1}{2(\kappa+1)}\sum_{i=1}^{m}x_i^2\Big\},\; -\infty \leq x_m \leq \ldots \leq x_1 < \infty \end{aligned} \tag{3.11}$$

$$\begin{aligned} f_1(\eta_{s+1},\ldots,\eta_{p_1}) &= 2^{(p_1-s) - \frac{1}{2}(p_1-s)(p_2-s)}\pi^{(p_1-s)/2}\Big\{\prod_{i=1}^{p_1-s}\Gamma\big(\tfrac{1}{2}(p_2-s-i+1)\big)\Gamma\big(\tfrac{1}{2}i\big)\Big\} \\ &\quad \times (\kappa+1)^{-(p_1-s)}\Big\{\prod_{i=s+1}^{p_1}\prod_{j=i+1}^{p_1}\frac{1}{\kappa+1}(\eta_i^2-\eta_j^2)\Big\}\Big(\prod_{i=s+1}^{p_1}\frac{\eta_i^2}{\kappa+1}\Big)^{\frac{1}{2}(p_2-p_1)} \\ &\quad \times \exp\Big(-\frac{1}{2(\kappa+1)}\sum_{i=s+1}^{p_1}\eta_i^2\Big),\; 0 \leq \eta_{p_1} \leq \ldots \leq \eta_{s+1} < \infty \end{aligned} \tag{3.12}$$

and

$$\kappa = \frac{\phi''(0)-[\phi'(0)]^2}{[\phi'(0)]^2}.$$

Proof. Most of the proof is similar to the proof given by Hsu (1941) for the case when the underlying distribution is normal.

(1) We find the limiting distribution of $r_{s+1},\ldots,r_{p_1}$, the (p_1-s) smallest canonical correlations.

We know that $r_1^2 \geq \cdots \geq r_{p_1}^2$ are positive roots of the determinantal equation

$$\begin{vmatrix} rS_{11} & S_{12} \\ S_{21} & rS_{22} \end{vmatrix} = 0. \tag{3.13}$$

Starting from (3.13), we obtain

$$|D-(\eta^2/(\kappa+1))I| = 0 \tag{3.14}$$

where $D = (d_{oj})$, and

$$d_{ij} = \sum_{g=s+1}^{p_2} \frac{v_{ig}v_{jg}}{\kappa+1} \quad i,j = s+1,\ldots,p_1 \tag{3.15}$$

Let $\zeta_{s+1} \geq \cdots \geq \zeta_{p_1}$ be the eigenvalues of matrix (d_{ij}). Then by Lemma 3.3 we have

$$n^{1/2}r_i \to [(\kappa+1)\zeta_i]^{1/2} \quad i = s+1,\ldots,p_1. \tag{3.16}$$

By Lemma 3.2 we know that the variables $\frac{v_{ig}}{\sqrt{\kappa+1}}$ are uncorrelated and asymptotically distributed as normal with mean zero and variance 1. Therefore the limiting distribution of the $\zeta_i = \frac{n\,r_i^2}{\kappa+1}$ has the density function

$$2^{-(p_1-s)(p_2-s)/2}\,\pi^{(p_1-s)/2}\,\{\prod_{i=1}^{p_1-s}\Gamma\{\tfrac{1}{2}(p_1-s-i+1)\Gamma\tfrac{1}{2}i\}\}^{-1}\{\prod_{i=s+1}^{p_1}\prod_{j=i+1}^{p_1}(\zeta_i-\zeta_j)\}$$

$$\times\{\prod_{i=s+1}^{p}\zeta_i\}^{(p_2-p_1-1)/2}\,\exp(-\tfrac{1}{2}\sum_{i=s+1}^{p_1}\zeta_i) \tag{3.17}$$

Since $\zeta_i = \frac{\eta_i^2}{\kappa+1}$, the density of the limiting distribution of the $\eta_i = n^{1/2} r_i$ is represented as (3.12).

We now find the limiting distribution of $r_1,\ldots,r_s$. From (3.13) we have

$$|S_{12}S_{22}^{-1}S_{21}-r^2S_{11}| = 0 \tag{3.18}$$

Setting $\theta = r^2$ and following the same lines as in Hsu (1941b) , we obtain, as $n\to\infty$,

$$\begin{vmatrix} F_{11}-\zeta & \cdots & F_{1\mu_1} \\ F_{\mu_1,1} & \cdots & F_{\mu_1,\mu_1}-\zeta \end{vmatrix} = 0 \tag{3.19}$$

where

$$F_{ij} = v_{ij} + v_{ji} - \rho_1(u_{ij}+w_{ij}) \quad i,j = 1,\ldots,\mu_1. \tag{3.20}$$

Let $\zeta_1' \geq \cdots \geq \zeta_{\mu_1}'$ be the roots of (3.19). Then by Lemma 3.3 we have

$$\frac{n^{1/2}(\theta_i - \rho_1^2)}{\rho_1} \to \zeta_i' \quad \text{as } n\to\infty \quad i = 1,\ldots,\mu_1 \tag{3.21}$$

From Lemma 3.2 we easily get that F's approach $\frac{1}{2}\mu_1(\mu_1+1)$ normal variates whose means are zero and second moments are as follows

$$\begin{aligned} &E\,(F_{ii}^2) = 4(\kappa+1)(1-\rho_1^2)^2 \\ &E\,(F_{ij}^2) = 2(\kappa+1)(1-\rho_1^2)^2 \qquad i \neq j \\ &E\,F_{ij}F_{k\ell} = 0 \quad i = k \text{ or } j \neq \ell \text{ or } i \neq k \text{ and } j \neq \ell. \end{aligned} \tag{3.22}$$

Hence the limiting distribution of ζ_i' may be derived as the distribution of the eigenvalues of the matrix (F_{ij}) in which the F's are regarded as normal variates. Let

$$F_{ii} = 2\sqrt{\kappa+1}\ (1-\rho_1^2)t_{ii}$$
$$F_{ij} = \sqrt{2(\kappa+1)}\ (1-\rho_1^2)t_{ij} \quad i \neq j, \tag{3.23}$$

the t's are mutually independent normal variates with zero mean and unit standard deviation.

Setting $\zeta = 2(1-\rho_1^2)\eta$ in (3.19) we get by (3.23)

$$\begin{vmatrix} t_{11}-(\kappa+1)^{-1/2}\eta & 2^{-1/2}t_{12} & \cdots & 2^{-1/2}t_{1,\mu_1} \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdots & \cdot\;\cdot\;\cdot \\ 2^{-1/2}t_{\mu_1,1} & 2^{-1/2}t_{\mu_1,2} & \cdots & t_{\mu_1\mu_1}-(\kappa+1)^{-1/2}\eta \end{vmatrix} = 0. \tag{3.24}$$

Let $\eta_1' \geq \ldots \geq \eta_{\mu_1}'$ be the roots of (3.24). Then we get the limiting distribution of $\eta_1', \ldots, \eta_{\mu_1}'$, from Hsu (1939), as

$$f(\eta_1', \ldots, \eta_{\mu_1}')$$
$$= 2^{-\mu_1(\mu_1+1)/2}\ (\kappa+1)^{-\mu_1/2}\ \left(\prod_{i=1}^{\mu_1} \Gamma\ \tfrac{1}{2} i\right)^{-1} \left\{\prod_{i=1}^{\mu_1} \prod_{j=i+1}^{\mu_1} \left(\frac{\eta_i'}{\sqrt{\kappa+1}} - \frac{\eta_j'}{\sqrt{\kappa+1}}\right)\right\}$$
$$\times \exp\left\{-\frac{1}{2(\kappa+1)} \sum_{i=1}^{\mu_1} \eta_i'^2\right\} \quad \infty > \eta_1' \geq \ldots \geq \eta_{\mu_1} > -\infty\ . \tag{3.25}$$

Similar to Hsu (1941b) if we define

$$r_i = \rho_1 + n^{-1/2}\ (1-\rho_1^2)\eta_i \quad i = 1, \ldots, \mu_1,$$

then

$$\eta_i = n^{1/2}(1-\rho_1^2)^{-1}(r_i-\rho_1)$$

have the same limiting distribution as the η_i'. Hence the limiting distribution of $\eta_1, \ldots, \eta_{\mu_1}$ has the density $f(\eta_1, \ldots, \eta_{\mu_1})$.

In exactly the same manner we may prove that for $\ell = 1,\ldots,s$ the

$$\eta_i = n^{1/2}(1-\rho_\ell^2)^{-1}(r_i-\rho_\ell)$$

$$i = \mu_1+\ldots+\mu_{\ell-1}+1,\ldots,\mu_1+\ldots+\mu_\ell$$

have the limiting distribution represented by the density

$$f(\eta_{\mu_1+\ldots+\mu_{\ell-1}+1},\ldots,\eta_{\mu_1+\ldots+\mu_\ell})$$

where, in general, $f(x_1,\ldots,x_m)$ is represented as (3.11).

It is easy to see that the sets $(\eta_1,\ldots,\eta_{\mu_1})$, $(\eta_{\mu_1+1},\ldots,\eta_{\mu_1+\mu_2}),\ldots,$ $(\eta_{\mu_1+\ldots+\mu_{s-1}+1},\ldots,\eta_{\mu_s})$ and $(\eta_{s+1},\ldots,\eta_p)$ are independent of one another. The proof is complete.

<u>Remark</u>. We may see from Theorem 3.1 that in the elliptical distributions family the limiting distribution of sample canonical correlations depend only on parameter κ. In the normal case $\kappa = 0$ and in this case Theorem 3.1 reduces to the result obtained by Hsu (1941b).

We will now derive the asymptotic joint distribution of the eigenvalues of the complex canonical correlation matrix under two situations. In the first situation, we assume that the density of Z is given by (2.11). As in the real case, we can show that the density of $W_{11}^{-1}W_{12}W_{22}^{-1}W_{21}$ is the same as when the underlying distribution is complex multivariate normal. In the second situation we assume that $\underset{\sim}{Z}_1,\ldots,\underset{\sim}{Z}_n$ are distributed independently and identically with a common density

$$f(\underset{\sim}{Z}) = 2^p \, |\Sigma|^{-1} h(2\, \bar{\underset{\sim}{Z}}'\Sigma^{-1}\underset{\sim}{Z}). \tag{3.26}$$

In this case, the characteristic function of $\underset{\sim}{Z}$ is given by $\phi(\bar{\underset{\sim}{t}}'\Sigma \underset{\sim}{t}/2)$ and the covariance matrix of $\underset{\sim}{Z}$ is $-2\phi'(0)\Sigma$. We can show that

$$E\{Z_j Z_k Z_\ell Z_m\} = 4\phi''(0)(\sigma_{jk}\sigma_{\ell m}+\sigma_{j\ell}\sigma_{km}+\sigma_{jm}\sigma_{k\ell}) \tag{3.27}$$

and the elements of the matrix W, after suitable standardization, are jointly distributed as complex multivariate normal.

Now, let $\lambda_1' \geq \ldots \geq \lambda_{p_1}'$ denote the eigenvalues of $\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ such that

$$\begin{aligned}
&\lambda_1' = \ldots = \lambda_{\mu_1}' = \lambda_1 \\
&\lambda_{\mu_1+1}' = \ldots = \lambda_{\mu_1+\mu_2}' = \lambda_2 \\
&\quad \vdots \\
&\lambda_{\mu_1+\ldots+\mu_{s-1}+1}' = \lambda_{\mu_1+\ldots+\mu_s}' = \lambda_s \\
&\lambda_{s+1}' = \ldots = \lambda_{p_1}' = 0
\end{aligned} \tag{3.28}$$

where $\lambda_1 > \ldots > \lambda_s$. Also, let

$$\delta_i = n^{1/2}(1-\lambda_i')^{-1}(\theta_i - \lambda_i') \tag{3.29}$$

for $i = 1,2,\ldots,p_1$. Also, let $\underset{\sim}{Z}_1,\ldots,\underset{\sim}{Z}_n$ be distributed independently and identically with a common density function given by (3.26). Then, following the same lines as in Theorem 3.1, we obtain the following:

<u>Theorem 3.2</u>. The joint asymptotic density function of $\delta_1,\ldots,\delta_{p_1}$ is given by

$$g(\delta_1,\ldots,\delta_{p_1}) = g(\delta_1,\ldots,\delta_{\mu_1})\ldots g(\delta_{\mu_1+\ldots+\mu_{s-1}+1},\ldots\delta_{\mu_1+\ldots+\mu_s})$$

$$\times\ g_1(\delta_{s+1},\ldots,\delta_{p_1}) \tag{3.30}$$

$$g(x_1,\ldots,x_m) = (\kappa+1)^{-m/2}\Big(\prod_{i=1}^{m}\Gamma(i)\Big)^{-1}\Big\{\prod_{i=1}^{m}\prod_{j=i+1}^{m}\Big(\frac{x_i}{\sqrt{\kappa+1}}-\frac{x_j}{\sqrt{\kappa+1}}\Big)\Big\}$$

$$\times\ \exp\{-\sum_{i=1}^{m}x_i^2/(\kappa+1)\} \tag{3.31}$$

$$g_1(\delta_{s+1},\ldots,\delta_{p_1}) = \pi^{p_1-s}\Big\{\prod_{i=1}^{p_1-s}\Gamma(p_2-s-i+1)\Gamma(i)\Big\}^{-1}(\kappa+1)^{-(p_1-s)}$$

$$\times\ \Big\{\prod_{i=s+1}^{p_1}\prod_{j=i+1}^{p_1}\{(\delta_i^2-\delta_j^2)/(\kappa+1)\}\Big\{\prod_{i=s+1}^{p_1}(\delta_i^2/\kappa+1)\Big\}^{p_2-p_1}$$

$$\times\ \exp\{-\sum_{i=s+1}^{p_1}\delta_i^2/(\kappa+1)\} \tag{3.32}$$

and

$$\kappa = \frac{\phi''(0)-\{\phi'(0)\}^2}{\{\phi'(0)\}^2}\ . \tag{3.33}$$

We will now discuss the relationship of the results of this section with the earlier work of Fang and Krishnaiah (1982), Fujikoshi (1978), Krishnaiah and Lee (1979), and Muirhead and Waternaux (1982) who used perturbation technique.

Let $f_i(r_1^2,\ldots,r_{p_1}^2)$, $(i=1,2,\ldots,k)$, be an analytic function of $r_1^2,\ldots,r_{p_1}^2$ around $\rho_1'^2,\ldots,\rho_{p_1}'^2$. In addition, let $L_i = \sqrt{n}\,\{T_i(\ell_1,\ldots,\ell_{p_1})-T_i(\lambda_1,\ldots,\lambda_{p_1})\}$,

$$\left.\frac{\partial T_i(r_1^2,\ldots,r_{p_1}^2)}{\partial r_{j_1}^2}\right|_{\underset{\sim}{\ell}=\underset{\sim}{\lambda}} = a_{i\alpha}$$

$$\left.\frac{\partial^2 T_i(r_1^2,\ldots,r_{p_1}^2)}{\partial r_{j_2}^2\,\partial r_{j_1}^2}\right|_{\underset{\sim}{\ell}=\underset{\sim}{\lambda}} = a_{i\alpha\beta} \tag{3.34}$$

for $j_1 \in J_\alpha$, $j_2 \in J_\beta$, $\underset{\sim}{\lambda} = (\lambda_1,\ldots,\lambda_{p_1})'$, $\underset{\sim}{\ell} = (\ell_1,\ldots,\ell_{p_1})'$, $\lambda_i = \rho_i'^2$, $\ell_i = r_i^2$ and J_α denotes the set of integers $\mu_1+\ldots+\mu_{\alpha-1}+1,\ldots,\mu_1+\ldots\mu_\alpha$ for $\alpha = 1,2,\ldots,k$.

When canonical correlations have multipicity as in (3.5), and the underlying distribution is multivariate normal, Fujikoshi (1978) derived asymptotic expression of T_1 whereas Krishnaiah and Lee (1979) derived joint asymptotic distribution of $T_1,\ldots,T_k$. In both of the above papers, it was implicitly assumed that none of the population canonical correlations are zero. If $\rho'_{s+1}=\ldots=\rho'_{p_1} = 0$, then the joint asymptotic distribution of $L_{10},\ldots,L_{k0}$ up to term of order $n^{-1/2}$ can be obtained by following the same lines as in Fang and Krishnaiah (1982) when the underlying distribution is nonnormal. Here

$$L_{i0} = \sqrt{n}\,\{T_i(r_1^2,\ldots,r_s^2)-T_i(\lambda_1,\ldots,\lambda_s)\}$$

for $i = 1,2,\ldots,k$. Now, let

$$c_i = \frac{\sqrt{n}(\ell_i-\lambda_i)}{2\rho_i'(1-\lambda_i)} \tag{3.35}$$

for $i = 1,2,\ldots,p_1$ when none of the population canonical correlations are zero. Muirhead and Waternaux (1980) showed that $c_1,\ldots,c_{p_1}$ are asymptotically distributed independently as normal with mean 0 and variance one when $\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n$ is a sample from a population with density (3.2) and the population canonical correlations are distinct. Fang and Krishnaiah (1982) derived the joint asymptotic distribution of $L_1,\ldots,L_k$ when the population canonical correlations are positive and satisfy the relation (3.5) and the underlying distribution is nonnormal. In their expression, the first term involves multivariate normal density whereas the second term (which is of order $n^{-1/2}$) involves multivariate Hermite polynomial. If we ignore the second term, a special case of the above expression is the result of Muirhead and Waternaux (1980).

Now, let us assume that the first s canonical correlations have multiplicity as in (3.5) and $\rho'_{s+1}=\ldots=\rho'_{p_1} > 0$ and $\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n$ is a sample from a population whose density function is given by (3.2). Then, the asymptotic joint distribution of $\eta_1,\ldots,\eta_{p_1}$ is obtained in the same way as the asymptotic joint distribution of $\eta_1,\ldots,\eta_s$ is derived in section 3. The joint asymptotic distribution of $\eta_1,\ldots,\eta_{p_1}$ is this case, is given by

$$f(\eta_1,\ldots,\eta_{\mu_1})f(\eta_{\mu_1+1},\ldots,\eta_{\mu_1+\mu_2})\ldots f(\eta_{s+1},\ldots,\eta_{p_1}) \tag{3.36}$$

where $f(x_1,\ldots,x_n)$ is given by (3.11). Now, let $T_i(\ell_1,\ldots,\ell_p)=\ell_i$. If $\mu_i > 1$ for at least one i, the asymptotic joint distribution of $c_1,\ldots,c_{p_1}$ cannot be obtained from the results of Fang and Krishnaiah (1982) since the assumption (3.34) is violated. But, the joint asymptotic distribution of $c_1^*,\ldots,c_s^*$, c_{s+1}^* follows from the result of Fang and Krishnaiah (1982) where

$$c_t^*=\frac{\sqrt{n}}{2\rho_t(1-\rho_t^2)\mu_t}\{(\ell_{\mu_1+\ldots+\mu_{t-1}+1}+\ldots+\ell_{\mu_1+\ldots+\mu_t})-(\lambda_{\mu_1+\ldots+\mu_{t-1}}+\ldots+\lambda_{\mu_1+\ldots+\mu_t})\} \tag{3.37}$$

The above result follows also from (3.36).

4. APPLICATIONS

In this section, we discuss procedures alternative to LRT procedure for testing the hypotheses on the ranks of the canonical correlation matrices in the real and complex cases.

We wish to test the hypothesis H_s against the alternative that H_s is not true. We can use a suitable function $\psi(r_{s+1}^2,\ldots,r_{p_1}^2)$ of $r_{s+1}^2,\ldots,r_{p_1}^2$ as test statistic to test for H_s. The hypothesis H_s is accepted or rejected according as

$$\psi(r_{s+1}^2,\ldots,r_{p_1}^2) \lessgtr c_\alpha \tag{4.1}$$

where

$$P[\psi(r_{s+1}^2,\ldots,r_{p_1}^2) \leq c_\alpha | H_s] = (1-\alpha). \tag{4.2}$$

We can choose $\psi(r_{s+1}^2,\ldots,r_{p_1}^2)$ as $r_{s+1}^2, r_{s+1}^2+\ldots+r_{p_1}^2$, etc. When H_s is true, the joint distribution of $n(r_{s+1}^2/\kappa+1,\ldots,r_{p_1}^2/\kappa+1)$ is the same as the joint distribution of the eigenvalues of the central Wishart matrix of order $(p_1-s)\times(p_1-s)$ and (p_2-s) degrees of freedom. So, we can get approximations to the percentage points of r_{s+1}^2 from the table of the distribution of the largest eigenvalue of the central Wishart matrix W_{p_1-s} and $E(W_{p_1-s}) = (p_2-s)I_{p_1-s}$. These tables are given in Krishnaiah (1980). Approximate percentage points associated with the test based upon $(r_{s+1}^2+..+r_{p_1}^2)$ can be obtained since $n(r_{s+1}^2+\ldots+r_{p_1}^2)/(\kappa+1)$ is asymptotically distributed as central chi-square with $(p_1-s)(p_2-s)$ degrees of freedom. We will now discuss a sequential procedure to determine the rank of the canonical correlation matrix.

The hypothesis H_0 is accepted or rejected according as

$$r_1^2 \lessgtr c_{1\alpha} \tag{4.3}$$

where

$$P[r_1^2 \le c_{1\alpha} | H_0] = (1-\alpha_1). \tag{4.4}$$

If H_0 is rejected, we accept or reject H_1 according as

$$r_2^2 \lessgtr c_{2\alpha} \tag{4.5}$$

where

$$P[r_2^2 \le c_{2\alpha} | r_1^2 \ge c_{1\alpha};\ H_1] = (1-\alpha_2). \tag{4.6}$$

When H_1 is true, $(nr_2^2/\kappa+1,\ldots,nr_{p_1}^2/\kappa+1)$ are asymptotically distributed independent of r_1^2 as the joint distribution of the eigenvalues of the central Wishart matrix $W(p_1-1;\ p_2-1;\ I)$; here $W(r,t;I)$ denotes the Wishart matrix of order $r\times r$ with t degrees of freedom and the expected value of the matrix is rI. If H_1 is accepted, we do not proceed further. If H_1 is rejected, we accept or reject H_2 according as

$$r_3^2 \lessgtr c_{3\alpha}$$

where

$$P[r_3^2 \le c_{3\alpha} | H_2; r_2^2 \ge c_{2\alpha}] = (1-\alpha_3). \tag{4.7}$$

When H_2 is true, $(nr_3^2/(\kappa+1),\ldots,nr_{p_1}^2/(\kappa+1))$ is asymptotically distributed independent of r_1^2 and r_2^2 and is the same as the joint distribution of the eigenvalues of the central Wishart matrix $W(p_1-2,p_2-2;I)$. We continue this procedure until a decision is made about the rank of the canonical correlation matrix.

When the underlying distribution is complex elliptical, we can propose test procedures on the rank of the canonical correlation matrix in the same way as above. We will now discuss applications of these procedures in the area of time series in the frequency domain.

Let $\underset{\sim}{X}'(t) = (\underset{\sim}{X}_1'(t),\underset{\sim}{X}_2'(t))$, $(t = 1,2,\ldots,T)$, be a $1\times u$ random vector which is distributed as a stationary Gaussian multivariate time series with zero mean vector and covariance matrix $R(v) = E\{\underset{\sim}{X}(t)\underset{\sim}{X}'(t+v)\}$. Also, let the spectral density matrix be denoted by $F(\omega)$

$$F(\omega) = (1/2\pi) \sum_{v=-\infty}^{\infty} \exp(-iv\omega) R(v) \tag{4.8}$$

where

$$F(\omega) = \begin{bmatrix} F_{11}(\omega) & F_{12}(\omega) \\ F_{21}(\omega) & F_{22}(\omega) \end{bmatrix} \tag{4.9}$$

and $F_{ij}(\omega)$ is of order $p_i \times p_j$. An estimate of $F(\omega)$ is known (e.g., see Parzen (1969) and Brillinger (1975)) to be $\hat{F}(\omega) = (\hat{f}_{jk}(\omega))$ where

$$\hat{f}_{jk}(\omega) = \frac{1}{(2m+1)} \sum_{a=-m}^{m} I_{jk}(\omega+(2\pi a/T)) \tag{4.10}$$

$$I_{jk}(\omega) = Z_j(\omega)\bar{Z}_k(\omega)$$

$$Z_j(\omega) = (1/(2\pi T)^{1/2}) \sum_{t=1}^{T} X(t)\exp(-it\omega)$$

and $\underset{\sim}{X}'(t) = (X_1(t),\ldots,S_p(t))$. It is known (see Goodman (1963) and Wahba (1968)) that $\hat{F}(\omega)$ is approximately distributed as complex Wishart matrix with (2m+1) degrees of freedom and $E\{\hat{F}(\omega)\} = (2m+1)F(\omega)$. So, we can test for the rank of $F_{12}(\omega)$ by following the methods described before in this paper.

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4. TITLE (and Subtitle) INFERENCE ON THE RANKS OF THE CANONICAL CORRELATION MATRICES FOR ELLIPTICALLY SYMMETRIC POPULATIONS		5. TYPE OF REPORT & PERIOD COVERED Technical - May 1985
		6. PERFORMING ORG. REPORT NUMBER 85-14
7. AUTHOR(s) P.R. Krishnaiah, J. Lin and L. Wang		8. CONTRACT OR GRANT NUMBER(s) F49620-85-C-0008
9. PERFORMING ORGANIZATION NAME AND ADDRESS Center for Multivariate Analysis Univ. of Pittsburgh, Fifth Floor, Thackeray Hall Pittsburgh, PA 15260		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research Department of the Air Force Bolling Air Force Base, DC 20332		12. REPORT DATE May 1985
		13. NUMBER OF PAGES 27
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Asymptotic distributions, complex distributions, canonical correlations, elliptical distribution, and time series.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, the authors considered the likelihood ratio tests and some other tests for the ranks of the canonical correlation matrices when the underlying distributions are real and complex elliptically symmetric distributions. Also, asymptotic joint distributions of the eigenvalues of the sample canonical correlation matrices are derived under the assumptions mentioned above regarding the underlying distributions. Finally, applications of tests for the rank of the complex canonical correlation matrix in the area of time series in the frequency domain are discussed.

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